

An application of the 3-dimensional q -deformed harmonic oscillator to the nuclear shell model

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Abstract

An analysis of the construction of a q -deformed version of the 3-dimensional harmonic oscillator, which is based on the application of q -deformed algebras, is presented. The results together with their applicability to the shell model are compared with the predictions of the modified harmonic oscillator.

1 Introduction

The assumption of an independent particle motion is the basis of shell model in finite nuclei. The average field is correctly described by the Woods-Saxon potential $V_{WS}(\mathbf{r})$. Since, however, the corresponding eigenvalue equation can be solved only numerically, it is desirable for many purposes to make use of the so called modified harmonic oscillator (MHO) Hamiltonian, which reproduces approximately the energy spectrum of the WS potential and at the same time has a simple analytical solution. The MHO was suggested by Nilsson [1] and studied in detail in [2, 3, 4]. It has the form

$$V_{MHO} = \frac{1}{2}\hbar\omega\rho^2 - \hbar\omega k \left\{ \mu \left(\mathbf{L}^2 - \langle \mathbf{L}^2 \rangle_N \right) + 2\mathbf{L}\cdot\mathbf{S} \right\}, \quad \rho = \sqrt{\frac{M\omega}{\hbar}} r \quad (1)$$

where \mathbf{L}^2 is the square of the angular momentum and $\mathbf{L}\cdot\mathbf{S}$ is the spin-orbit interaction.

As it is well known, the subtraction of the average value of $\langle \mathbf{L}^2 \rangle$, taken over each N -shell

$$\langle \mathbf{L}^2 \rangle_N = \frac{N(N+3)}{2} \quad (2)$$

avoids the shell compression induced by the \mathbf{L}^2 term, leaving the “center of gravity” of each N -shell unchanged. The V_{MHO} (without the spin-orbit term $\mathbf{L}\cdot\mathbf{S}$) reproduces effectively

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the Woods - Saxon radial potential. A better agreement with the experimental data can be achieved if the parameters of the potential are let to vary smoothly from shell to shell.

It should be noted, however, that from mathematical point of view the introduction of \mathbf{L}^2 -dependent term (and the corresponding correction $\langle \mathbf{L}^2 \rangle_N$) in the modified harmonic oscillator potential is not so innocent, as it look at first sight, because this term depends on the state in which the particle occurs, and in this sense the potential (1) is “non-local” and “deformable” (with variable deformation). This effect is amplified by the fact, that the constant $\mu' = k\mu$ varies from shell to shell.

The inclusion of \mathbf{L}^2 -dependent term together with the corresponding correction $\langle \mathbf{L}^2 \rangle_N$ and of the N -dependence of μ' can be considered as a first step for the introduction of some non-trivial “ deformations” into the Hamiltonian of the 3-dimensional harmonic oscillator. There is, however, an alternative approach to the description of the space extended and deformable many body systems based on the application of the q -deformed algebras.

In nuclear physics, the $su_q(2)$ quantum algebra has been used with promising results for the description of rotational bands in the deformed and superdeformed nuclei [5] - [15]. Symmetries based on quantum algebras have been adopted successfully for studying rotational [16, 17], vibrational [18, 19] and rotational - vibrational spectra [20]-[22] of diatomic molecules. Quantum groups extensions of some simple nuclear models [23]-[29] have also been shown to lead to interesting results.

The aim of this paper is to construct a Hamiltonian of the q -deformed 3-dimensional harmonic oscillator in terms of the q -deformed boson operators [30, 31]. It will be shown that, in the $q \rightarrow 1$ limit, its eigenvalues qE coincide with the eigenvalues of the modified harmonic oscillator (MHO) without a spin-orbit term. More specifically, we represent q in the form $q = e^\tau$ and show that the expansion of qE in series of τ yields a Hamiltonian having the same structure as the one given by (1), with τ playing the role of $\mu' = k\mu$. It may be worth to stress at this point, that the construction of the Hamiltonian ${}^qH_{3dim}$ is a non-trivial problem since, from a physical point of view, it is necessary to separate explicitly the q -deformed angular momentum term ${}^q\mathbf{L}^2$.

In Section 2 we introduce q -deformed $so_q(3)$ generators (q -analogs of the angular momentum operators), which act in the space of the most symmetric representation of the algebra $u_q(3)$, which, as a matter of fact, is the space of the q -deformed 3-dimensional harmonic oscillator [32, 33, 34].

In Section 3 we suggest an expression for the Hamiltonian of the q -deformed 3-dimensional harmonic oscillator, which is invariant with respect to the q -deformed angular momentum algebra. The vibrational and rotational degrees of freedom of this Hamiltonian are clearly separated, which is in agreement with the classical situation and in the limit $q \rightarrow 1$ it coincides with the classical Hamiltonian (without the correction term (2)). In the case of small deformations of the algebra its spectrum reproduces very well the spectrum of the 3-dimensional MHO with the correction (2).

The next step (Section 4) is to introduce the q -deformed spin-orbit term $\mathcal{L}^{(q)} \cdot \mathcal{S}^{(1/q)}$ and to show that in the case of a small deformations this term introduces small, but essential corrections to the standard spin-orbit term $\mathbf{L} \cdot \mathbf{S}$.

2 $so_q(3)$ -algebra and $so_q(3)$ -basis states

We shall use the three independent q -deformed boson operators b_i and b_i^\dagger ($i = +, 0, -$), which satisfy the commutation relations [30, 31]

$$[N_i, b_i^\dagger] = b_i^\dagger \quad [N_i, b_i] = -b_i \quad b_i b_i^\dagger - q^{\pm 1} b_i^\dagger b_i = q^{\mp N_i} \quad (3)$$

where N_i are the corresponding number operators.

The elements of the $so_q(3)$ algebra (i.e. the angular momentum operators) acting in the Fock space of the totally symmetric representations $[N, 0, 0]$ of $u_q(3)$ have been derived by Van der Jeugt [32, 33, 34]. A simplified form of the subalgebra $so_q(3) \subset u_q(3)$, can be obtained by introducing the operators [36]

$$\begin{aligned} B_0 &= q^{-\frac{1}{2}N_0} b_0 & B_0^\dagger &= b_0^\dagger q^{-\frac{1}{2}N_0} \\ B_i &= q^{N_i+\frac{1}{2}} b_i \sqrt{\frac{[2N_i]}{[N_i]}} & B_i^\dagger &= \sqrt{\frac{[2N_i]}{[N_i]}} b_i^\dagger q^{N_i+\frac{1}{2}} \quad i = +, - \end{aligned} \quad (4)$$

where the q -numbers $[x]$ are defined as

$$[x] = (q^x - q^{-x}) / (q - q^{-1})$$

and we shall consider only real values for the deformation parameter q . These new operators satisfy the usual commutation relations

$$[N_i, B_i^\dagger] = B_i^\dagger \quad [N_i, B_i] = -B_i. \quad (5)$$

In the Fock space, spanned on the normalized eigenvectors of the excitation number operators N_+, N_0, N_- , they satisfy the relations

$$\begin{aligned} B_0^\dagger B_0 &= q^{-N_0+1} [N_0] & B_0 B_0^\dagger &= q^{-N_0} [N_0 + 1] \\ B_i^\dagger B_i &= q^{2N_i-1} [2N_i] & B_i B_i^\dagger &= q^{2N_i+1} [2N_i + 2] \quad i = +, - \end{aligned} \quad (6)$$

and hence, the commutation relations

$$[B_0, B_0^\dagger] = q^{-2N_0} \quad [B_i, B_i^\dagger] = [2]q^{4N_i+1} \quad i = +, - \quad (7)$$

It was shown in [36] that the angular momentum operators defined in [32, 33], when are expressed in terms of the modified operators (4), take the simplified form

$$\begin{aligned} L_0 &= N_+ - N_- \\ L_+ &= q^{-L_0+\frac{1}{2}} B_+^\dagger B_0 + q^{L_0-\frac{1}{2}} B_0^\dagger B_- \\ L_- &= q^{-L_0-\frac{1}{2}} B_0^\dagger B_+ + q^{L_0+\frac{1}{2}} B_-^\dagger B_0. \end{aligned} \quad (8)$$

and satisfy the standard $so_q(3)$ commutation relations

$$[L_0, L_\pm] = \pm L_\pm \quad [L_+, L_-] = [2L_0] \quad (9)$$

As it has been discussed in detail in [32] the algebra (8) is not a subalgebra of $u_q(3)$ considered as Hopf algebras, but as q -deformed enveloping algebras. However, in our context, working in the Fock space (constructed from three independent q -bosons), we

restrict ourselves to the symmetric representation of $u_q(3)$, where the embedding $so_q(3) \subset u_q(3)$ is valid.

The Casimir operator of $so_q(3)$ can be written in the form [37]

$$\begin{aligned} C_2^{(q)} &= \frac{1}{2} \{L_+ L_- + L_- L_+ + [2][L_0]^2\} \\ &= L_- L_+ + [L_0][L_0 + 1] = L_+ L_- + [L_0][L_0 - 1]. \end{aligned} \quad (10)$$

One can also define q -deformed states $|n\ell m\rangle_q$ satisfying the eigenvalue equations

$$\begin{aligned} {}^q\mathbf{L}^2 |n\ell m\rangle_q &= [\ell][\ell + 1] |n\ell m\rangle_q \\ L_0 |n\ell m\rangle_q &= m |n\ell m\rangle_q \\ N |n\ell m\rangle_q &= n |n\ell m\rangle_q \end{aligned} \quad (11)$$

Here ${}^q\mathbf{L}^2 \stackrel{def}{=} C_2^{(q)}$ is the “square” of the q -deformed angular momentum and $N = N_+ + N_0 + N_-$ is the total number operator for the q -deformed bosons. These states have the form

$$\begin{aligned} |n\ell m\rangle_q &= q^{\frac{1}{4}(n-\ell)(n+\ell+1)-\frac{1}{2}m^2} \sqrt{\frac{[n-\ell]!![\ell+m]![\ell-m]![2\ell+1]}{[n+\ell+1]!!}} \times \\ &\times \sum_{t=0}^{(n-\ell)/2} \sum_{p=\max(0,m)}^{[(\ell+m)/2]} \frac{(-1)^t q^{-(n+\ell+1)t}}{[2t]!![n-\ell-2t]!!} \frac{(B_+^\dagger)^{p+t}}{[2p]!!} \frac{(B_0^\dagger)^{n+m-2p-2t}}{[n+m-2p]!} \frac{(B_-^\dagger)^{p+t-m}}{[2p-2m]!!} |0\rangle \end{aligned} \quad (12)$$

where $|0\rangle$ is the vacuum state, $[n]! = [n][n-1]\dots[1]$, $[n]!! = [n][n-2]\dots[2]$ or $[1]$, and, as shown in [32, 33], form a basis for the most symmetric representation $[n, 0, 0]$ of $u_q(3)$, corresponding to the $u_q(3) \supset so_q(3)$ chain.

3 $so_q(3)$ vector operators and the Hamiltonian of the 3-dimensional q -deformed harmonic oscillator

Here we shall recall some well known definitions about the q -deformed tensor operators within the framework of the algebra $so_q(3)$. An irreducible tensor operator of rank j with parameter q according to the algebra $so_q(3)$ is a set of $2j+1$ operators $\mathcal{T}_{jm}^{(q)}$, satisfying the relations

$$\begin{aligned} [L_0, \mathcal{T}_{jm}^{(q)}] &= m \mathcal{T}_{jm}^{(q)} \\ [L_\pm, \mathcal{T}_{jm}^{(q)}]_{q^m} q^{L_0} &= \sqrt{[j \mp m][j \pm m + 1]} \mathcal{T}_{j, m \pm 1}^{(q)} \end{aligned} \quad (13)$$

where, in order to express the adjoint action of the generators L_0, L_\pm of $so_q(3)$ on the components of the tensor operator $\mathcal{T}_{jm}^{(q)}$, we use the usual notation of the q -commutator

$$[A, B]_{q^\alpha} = AB - q^\alpha BA. \quad (14)$$

By $\tilde{\mathcal{T}}_{jm}^{(q)}$ we denote the conjugate irreducible q -tensor operator

$$\tilde{\mathcal{T}}_{jm}^{(q)} = (-1)^{j-m} q^{-m} \mathcal{T}_{j, -m}^{(q)}, \quad (15)$$

which satisfies the relations

$$\begin{aligned} [\tilde{\mathcal{T}}_{jm}^{(q)}, L_0] &= m \tilde{\mathcal{T}}_{jm}^{(q)} \\ q^{L_0} [\tilde{\mathcal{T}}_{jm}^{(q)}, L_{\pm}] q^{-m} &= \sqrt{[j \pm m][j \mp m + 1]} \tilde{\mathcal{T}}_{j, m \mp 1}^{(q)}. \end{aligned} \quad (16)$$

Then the operator

$$\mathcal{P}_{jm}^{(q)} = (\tilde{\mathcal{T}}_{jm}^{(q)})^{\dagger} = (-1)^{j-m} q^{-m} (\mathcal{T}_{j, -m}^{(q)})^{\dagger} \quad (17)$$

where \dagger denotes hermitian conjugation, transforms in the same way (13) as the tensor $\mathcal{T}_{jm}^{(q)}$, i.e. $\mathcal{P}_{jm}^{(q)}$ also is an irreducible $so_q(3)$ tensor operator of rank j .

Let $\mathcal{A}_{j_1 m_1}^{(q_1)}$ and $\mathcal{B}_{j_2 m_2}^{(q_2)}$ be two irreducible tensor operators. We shall define the tensor and scalar product of these tensor operators following some of the prescriptions, summarized, for example, in [37]. One can introduce the following operator

$$[\mathcal{A}_{j_1}^{(q_1)} \times \mathcal{B}_{j_2}^{(q_2)}]_{jm}^{(q_3)} = \sum_{m_1, m_2} q_3 C_{j_1 m_1, j_2 m_2}^{jm} \mathcal{A}_{j_1 m_1}^{(q_1)} \mathcal{B}_{j_2 m_2}^{(q_2)} \quad (18)$$

where $q_3 C_{j_1 m_1, j_2 m_2}^{jm}$ are the Clebsch-Gordan coefficients corresponding to the deformation parameter q_3 . In general the deformation parameters q_1, q_2 and q_3 can be arbitrary. It turns out, however, if one imposes the condition that the left hand side of (18) transforms as an irreducible q -tensor of rank j in a way, that the Wigner-Eckhart theorem can be applied to (18) as a whole, not all of the combinations of q_1, q_2 and q_3 are allowed.

If the tensors $\mathcal{A}_{j_1}^{(q_1)}$ and $\mathcal{B}_{j_2}^{(q_2)}$ depend on one and same variable and act on a single vector, which depend on the same variable, the mentioned requirement will be satisfied only if $q_1 = q_2 = q$ and $q_3 = 1/q$, i.e. the operator

$$[\mathcal{A}_{j_1}^{(q)} \times \mathcal{B}_{j_2}^{(q)}]_{jm}^{(1/q)} = \sum_{m_1, m_2} 1/q C_{j_1 m_1, j_2 m_2}^{jm} \mathcal{A}_{j_1 m_1}^{(q)} \mathcal{B}_{j_2 m_2}^{(q)} \quad (19)$$

transforms as an irreducible q -tensor of rank j according to the algebra $so_q(3)$. Then, the definition (19) is in agreement with the property

$$\begin{aligned} \langle \alpha', \ell' | [\mathcal{A}_{j_1}^{(q)} \times \mathcal{B}_{j_2}^{(q)}]_j^{(1/q)} | \alpha, \ell \rangle &= \sqrt{[2j+1]} \sum_{\alpha'', \ell''} (-1)^{\ell+j+\ell'} \left\{ \begin{matrix} \ell & j & \ell' \\ j_1 & \ell'' & j_2 \end{matrix} \right\}_q \times \\ &\times \langle \alpha', \ell' | \mathcal{A}_{j_1}^{(q)} | \alpha'', \ell'' \rangle \langle \alpha'', \ell'' | \mathcal{B}_{j_2}^{(q)} | \alpha, \ell \rangle \end{aligned} \quad (20)$$

which is a q -analogue of the well known classical identity.

In Section 4 we shall consider also the scalar product of two tensor operators depending on two different variables (1) and (2) and acting on different vectors, depending on these different variables. In this particular case the *scalar product* of irreducible q and q^{-1} - tensor operators $\mathcal{A}_j^{(q)}(1)$, $\mathcal{B}_j^{(1/q)}(2)$ with the same rank, acting on *different vectors* (1) and (2), will be given by means of the following definition

$$\begin{aligned} (\mathcal{A}_j^{(q)}(1) \cdot \mathcal{B}_j^{(1/q)}(2))^q &= (-1)^j \sqrt{[2j+1]} [\mathcal{A}_j^{(q)}(1) \times \mathcal{B}_j^{(1/q)}(2)]_{00}^{(q)} \\ &= \sum_m (-q)^m \mathcal{A}_{jm}^{(q)}(1) \mathcal{B}_{j-m}^{(1/q)}(2). \end{aligned} \quad (21)$$

It should be noted that the angular momentum operators (8) satisfy the commutation relations (9), but *are not* tensor operators in the sense of definition (13). One can form, however, the angular momentum operators

$$\begin{aligned}\mathcal{L}_{\pm 1}^{(q)} &= \mp \frac{1}{\sqrt{[2]}} q^{-L_0} L_{\pm} \\ \mathcal{L}_0^{(q)} &= \frac{1}{[2]} \left\{ q[2L_0] + (q - q^{-1})L_-L_+ \right\} = \\ &= \frac{1}{[2]} \left\{ q[2L_0] + (q - q^{-1}) \left(C_2^{(q)} - [L_0][L_0 + 1] \right) \right\}\end{aligned}\quad (22)$$

which are tensors of 1-st rank, i.e $so_q(3)$ vectors¹ (see for example [37]).

One can easily check that the operators $B_0^\dagger, B_\pm^\dagger$ and B_0, B_\pm do not form a spherical vector. As shown in [36], one can define a vector operator T_m^\dagger of the form

$$\begin{aligned}T_{+1}^\dagger &= \frac{1}{\sqrt{[2]}} B_+^\dagger q^{-2N_+ + N - \frac{1}{2}} \\ T_0^\dagger &= B_0^\dagger q^{-2N_+ + N} \\ T_{-1}^\dagger &= \frac{1}{\sqrt{[2]}} \left\{ B_-^\dagger q^{2N_+ - N - \frac{1}{2}} - (q - q^{-1})B_+(B_0^\dagger)^2 q^{-2N_+ + N + \frac{3}{2}} \right\}.\end{aligned}\quad (23)$$

The corresponding expressions for the conjugate operators $\tilde{T}_m = (-1)^m q^{-m} (T_{-m}^\dagger)^\dagger$ are

$$\begin{aligned}\tilde{T}_{+1} &= -\frac{1}{\sqrt{[2]}} \left\{ q^{2N_+ - N - \frac{3}{2}} B_- - (q - q^{-1})q^{-2N_+ + N + \frac{1}{2}} B_+^\dagger (B_0)^2 \right\} \\ \tilde{T}_0 &= q^{-2N_+ + N} B_0 \\ \tilde{T}_{-1} &= -\frac{1}{\sqrt{[2]}} q^{-2N_+ + N + \frac{1}{2}} B_+.\end{aligned}\quad (24)$$

One can easily check, that vector operators \mathbf{T}^\dagger and $\tilde{\mathbf{T}}$ satisfy the commutation relation

$$\begin{aligned}[\tilde{T}_{-1}, T_{+1}^\dagger]_{q^{-2}} &= -q^{2N+1} \\ [\tilde{T}_0, T_0^\dagger] &= q^{2N} + q^{-1}(q^2 - q^{-2})T_{+1}^\dagger \tilde{T}_{-1} \\ [\tilde{T}_{+1}, T_{-1}^\dagger]_{q^{-2}} &= -q^{2N-1} + q^{-1}(q^2 - q^{-2})\{T_0^\dagger \tilde{T}_0 + (q - q^{-1})T_{+1}^\dagger \tilde{T}_{-1}\}\end{aligned}\quad (25)$$

and

$$\begin{aligned}[\tilde{T}_0, T_{+1}^\dagger] &= 0 & [\tilde{T}_{-1}, T_0^\dagger] &= 0 \\ [\tilde{T}_{+1}, T_{+1}^\dagger]_{q^2} &= 0 & [\tilde{T}_{-1}, T_{-1}^\dagger]_{q^2} &= 0 \\ [\tilde{T}_{+1}, T_0^\dagger] &= (q^2 - q^{-2})T_{+1}^\dagger \tilde{T}_0 & [\tilde{T}_0, T_{-1}^\dagger] &= (q^2 - q^{-2})T_0^\dagger \tilde{T}_{-1}\end{aligned}\quad (26)$$

¹ The situation in the “standard” theory of angular momentum is analogous. Indeed, from the operators $L_0 = L_3, L_\pm = L_1 \pm iL_2$, which are not spherical tensors, one can construct the operators $J_\pm = \mp \frac{1}{\sqrt{2}}(L_1 \pm iL_2), J_0 = L_0$, which are the components of a spherical tensor of 1-st rank according the standard $so(3)$ -algebra.

Unlike operators (4), the commutators $[\tilde{T}_m, \tilde{T}_n]$ and $[T_m^\dagger, T_n^\dagger]$ do not vanish, but are equal to

$$\begin{aligned} [T_{+1}^\dagger, T_0^\dagger]_{q^2} &= 0 & [\tilde{T}_0, \tilde{T}_{-1}]_{q^2} &= 0 \\ [T_0^\dagger, T_{-1}^\dagger]_{q^2} &= 0 & [\tilde{T}_{+1}, \tilde{T}_0]_{q^2} &= 0 \\ [T_{+1}^\dagger, T_{-1}^\dagger] &= (q - q^{-1})(T_0^\dagger)^2 & [\tilde{T}_{+1}, \tilde{T}_{-1}] &= (q - q^{-1})(\tilde{T}_0)^2 \end{aligned} \quad (27)$$

which is in agreement with the results obtained in [35].

It should be noted that the angular momentum operator $\mathcal{L}_M^{(q)}$ (considered as a vector according $so_q(3)$) can be represented in the form

$$\mathcal{L}_M^{(q)} = -\sqrt{\frac{[4]}{[2]}} [T^\dagger \times \tilde{T}]_{1M}^{(1/q)} = -\sqrt{\frac{[4]}{[2]}} \sum_{m,n}^{1/q} C_{1m,1n}^{1M} T_m^\dagger \tilde{T}_n \quad (28)$$

Its “square” differs from $C_2^{(q)}$ and equals to [37]

$$\begin{aligned} (\mathcal{L}^{(q)})^2 \equiv \mathcal{L}^{(q)} \cdot \mathcal{L}^{(q)} &= \sum (-q)^{-M} \mathcal{L}_M^{(q)} \mathcal{L}_{-M}^{(q)} \\ &= \frac{2}{[2]} C_2^{(q)} + \left(\frac{q - q^{-1}}{[2]} \right)^2 (C_2^{(q)})^2 \end{aligned} \quad (29)$$

This difference, however, is not essential. Let us consider, for instance, the expectation values of the scalar operator (29). It has the form

$${}_q \langle \ell m | \mathcal{L}^{(q)} \cdot \mathcal{L}^{(q)} | \ell m \rangle_q = \frac{[2\ell][2\ell + 2]}{[2]^2} = [\ell]_{q^2} [\ell + 1]_{q^2} \quad (30)$$

In this sense the replacement of ${}^q \mathbf{L}^2 \equiv C_2^{(q)}$ with $\mathcal{L}^{(q)} \cdot \mathcal{L}^{(q)}$ is equivalent to the replacement $q \rightarrow q^2$ and leads to the renormalization of some constant. For this reason we shall accept as the square of the *physical* angular momentum the quantity ${}^q \mathbf{L}^2 \equiv C_2^{(q)}$, whose eigenvalues are $[\ell][\ell + 1]$.

Now let us consider the scalar operator, constructed in terms of T_M^\dagger and \tilde{T}_M . We have

$$\begin{aligned} X_0^{(q)} &= -\sqrt{[3]} [T^\dagger \times \tilde{T}]_{00}^{(1/q)} \\ &= -q^{-1} T_{+1}^\dagger \tilde{T}_{-1} + T_0^\dagger \tilde{T}_0 - q T_{-1}^\dagger \tilde{T}_1 \end{aligned} \quad (31)$$

We define the Hamiltonian of the three dimensional q -deformed oscillator as

$${}^q H_{3dim} = \hbar \omega_0 X_0 = \hbar \omega_0 \left(-q^{-1} T_{+1}^\dagger \tilde{T}_{-1} + T_0^\dagger \tilde{T}_0 - q T_{-1}^\dagger \tilde{T}_1 \right) \quad (32)$$

The motivations for such an ansatz are:

1. The operator so defined is an $so_q(3)$ scalar, i.e. it is simultaneously measurable with the physical q -deformed angular momentum square ${}^q \mathbf{L}^2$ and the z-projection L_0 ;
2. Only this $so_q(3)$ -scalar has the property of being the sum of terms, each containing an equal number of creation and annihilation operators (i.e. conserves the number of bosons);
3. In the limit $q \rightarrow 1$ (32) goes into

$$\lim_{q \rightarrow 1} {}^q H_{3dim} = \hbar \omega_0 (a_{+1}^\dagger a_{+1} + a_0^\dagger a_0 + a_{-1}^\dagger a_{-1})$$

where $[a_m, a_n^\dagger] = \delta_{mn}$, i.e. in this limit (32) coincides with the Hamiltonian of the 3-dimensional (spherically symmetric) harmonic oscillator up to an additive constant. It should be underlined that the Hamiltonian (32) does not commute with the square of the “classical” (or “standard”) angular momentum \mathbf{L}^2 . This is due to the fact that the q -deformed oscillator is really space deformed and the “standard” quantum number l is not a good quantum number for this system. On the other hand ${}^q\mathbf{L}^2$ commutes with ${}^qH_{3dim}$ and the “quantum angular momentum” ℓ can be used for the classification of the states of the q -oscillator. However the projection of the standard angular momentum on the z axis l_z coincides with the quantum projectoin ℓ_z , i.e. it is a good quantum number.

Equation (32) can be cast in a simpler and physically more transparent form. Indeed, making use of the third component of the q -deformed angular momentum (considered as $so_q(3)$ vector)

$$\begin{aligned}\mathcal{L}_0^{(q)} &= -\sqrt{[4]/[2]} [T^\dagger \times \tilde{T}]_{10}^{(1/q)} \\ &= -T_{+1}^\dagger \tilde{T}_{-1} + (q - q^{-1}) T_0^\dagger \tilde{T}_0 + T_{-1}^\dagger \tilde{T}_0\end{aligned}\quad (33)$$

we obtain

$$X_0^{(q)} + q\mathcal{L}_0^{(q)} = -[2]T_{+1}^\dagger \tilde{T}_{-1} + q^2 T_0^\dagger \tilde{T}_0 \quad (34)$$

Since

$$T_{+1}^\dagger \tilde{T}_{-1} = \frac{[2N_+]}{[2]} q^{-2N_+ + 2N + 1}, \quad T_0^\dagger \tilde{T}_0 = [N_0] q^{-4N_+ - N_0 + 2N - 1} \quad (35)$$

we get upon summation

$$X_0^{(q)} = -q\mathcal{L}_0^{(q)} + [N + L_0] q^{N - L_0 + 1} \quad (36)$$

and, after some calculations,

$${}^qH_{3dim} = \hbar\omega_0 X_0^{(q)} = \hbar\omega_0 \left\{ [N] q^{N+1} - \frac{q(q - q^{-1})}{[2]} C_2^{(q)} \right\} \quad (37)$$

The eigenvalues of such a q -deformed Hamiltonian are

$${}^qE_{3dim} = \hbar\omega_0 \left\{ [n] q^{n+1} - \frac{q(q - q^{-1})}{[2]} [\ell][\ell + 1] \right\}, \quad \ell = n, n - 2, \dots, 0 \text{ or } 1 \quad (38)$$

In the $q \rightarrow 1$ limit we have $\lim_{q \rightarrow 1} {}^qE_{3dim} = \hbar\omega_0 n$, which coincides with the classical result. We shall note that, the expression (37) can also be represented in the form

$${}^qE_{3dim} = \hbar\omega_0 \frac{q^{n+1}}{[2]} \left\{ q^{\ell+1} [n - \ell] + q^{-\ell-1} [n + \ell] \right\} \quad (39)$$

but the advantage of the form (38) is that, the vibrational ($[n]$) and the rotational ($[\ell][\ell + 1]$) degrees of freedom are clearly separated.

For small values of the deformation parameter τ , where $q = e^\tau$, we can expand (38) in powers of τ obtaining

$$\begin{aligned}{}^qE_{3dim} &= \hbar\omega_0 n - \hbar\omega_0 \tau (\ell(\ell + 1) - n(n + 1)) \\ &\quad - \hbar\omega_0 \tau^2 \left(\ell(\ell + 1) - \frac{1}{3} n(n + 1)(2n + 1) \right) + \mathcal{O}(\tau^3)\end{aligned}\quad (40)$$

To leading order in τ the expression (40) closely resembles the one giving the energy eigenvalue of the MHO² (1) (if one neglects the spin-orbit term). Quantitatively, one obtains a good fit of the energy spectrum produced by the MHO potential (1) if we choose for $\hbar\omega_0$ and τ of the q -HO the values $\hbar\omega_0 = 0.94109$ and $\tau = 0.021948$ (we have assumed $\hbar\omega = 1$ for MHO and *neglected* the spin-orbit term). The value of τ is close to those adopted for $\mu' = k\mu$. Indeed the MHO fit in the $^{208}_{82}Pb$ region yields [3, 4] $\mu' = 0$ for $N = 2$, $\mu' = 0.0263$ for $N = 3$, $\mu' = 0.024$ for $N \geq 4$ and for k , fixed by the condition that the observed order of sub-shells be reproduced, $k = 0.08$ for $N = 2$, $k = 0.075$ for $N = 3$ and $k = 0.07$ for $N \geq 4$.

It appears surprising at first that $\hbar\omega_0$ differs slightly from 1. Indeed the correction term $n(n+1)$ in the q -HO spectrum is slightly different from the corresponding piece $n(n+3)/2$ in the MHO. Some compression of the q -HO spectrum is therefore unavoidable. From a physical point of view, one may say that the mean radius of the deformed oscillator is slightly larger than the radius of the classical isotropic HO.

MHO and q -deformed 3-dim HO spectra are compared in figure 1. It should be noted that the constant μ' in the modified harmonic oscillator potential takes different values for shells with different values of N . In the q -deformed model the parameters τ and $\hbar\omega_0$ have the same values for all the shells. The comparison shows that the q -deformation of the 3-dimensional harmonic oscillator effectively reproduces the non-locality and the “deformations” induced in the MHO-model through the terms \mathbf{L}^2 , $\langle \mathbf{L}^2 \rangle_N$ and the variability of μ' with the shell number N .

4 q -deformed 3-dimensional harmonic oscillator with q -deformed spin-orbit term

For a full comparison with the MHO and a more realistic description of the single-particle spectrum, we need to include a q -deformed spin-orbital term $\mathcal{L}^{(q)} \cdot \mathcal{S}^{(1/q)}$ in the q -deformed harmonic oscillator Hamiltonian ${}^qH_{3dim}$. To this purpose we shall introduce the spin operators S_+, S_0, S_- , which are elements of another (independent) $su_q(2)$ algebra. These operators satisfy the commutation relation (9), i.e.

$$[S_0, S_{\pm}] = \pm S_{\pm} \quad [S_+, S_-] = [2S_0] \quad (41)$$

and act in the *two-dimensional* representation space of this algebra. The orthonormalized basis vectors of this space will be denoted as always by $|\frac{1}{2}m_s\rangle_q$.

We now define the q -deformed total angular momentum as [37]

$$J_0 = L_0 + S_0 \quad (42)$$

$$J_{\pm} = L_{\pm}q^{S_0} + S_{\pm}q^{-L_0} \quad (43)$$

where the q -deformed orbital angular momentum is given by (8). The operators (42-43) satisfy the same commutation relations as the operators of the q -deformed orbital angular

² Indeed, these eigenvalues are

$$E_{nl} = \hbar\omega n - \hbar\omega\mu' \left(l(l+1) - \frac{1}{2}n(n+3) \right)$$

where μ' is allowed to vary from shell to shell.

momentum (9) and spin (41), and the corresponding expression for the Casimir operator of the algebra (42-43) can be written in the form

$$C_{2,J}^{(q)} = J_- J_+ + [J_0][J_0 + 1] \quad (44)$$

As in the case of the q -deformed orbital momentum we shall consider ${}^q J^2 \equiv C_{2,J}^{(q)}$ as the total angular momentum “square”.

The common eigenvectors of $C_{2,J}^{(q)}$, $C_{2,L}^{(q)}$ and $C_{2,S}^{(q)}$ can be written in the usual form

$$|n(\ell \frac{1}{2})jm\rangle_q = \sum_{m_\ell, m_s} {}^q C_{\ell m_\ell, \frac{1}{2} m_s}^{jm} |n\ell m_\ell\rangle_q |\frac{1}{2} m_s\rangle_q \quad (45)$$

where ${}^q C_{\ell m_\ell, \frac{1}{2} m_s}^{jm}$ are the q -deformed Clebsch-Gordan coefficients.

We define the spin-orbital term as a scalar product, according to the definition (21)

$$\begin{aligned} \mathcal{L}^{(q)} \cdot \mathcal{S}^{(1/q)} &= \sum_{M=0, \pm 1} (-q)^M \mathcal{L}_M^{(q)} \mathcal{S}_{-M}^{(1/q)} = \\ &= \frac{1}{[2]} \left\{ C_{2,J}^{(q)} - C_{2,L}^{(q)} - C_{2,S}^{(q)} - \frac{(q - q^{-1})^2}{[2]} C_{2,L}^{(q)} C_{2,S}^{(q)} \right\} \end{aligned} \quad (46)$$

where $\mathcal{S}^{(1/q)}$ is a vector operator according to the algebra (41) and it is constructed from the q -deformed spin operators taking into account the rule (22), but for deformation parameter $1/q$. The Hamiltonian, which we suggest for the q -deformed 3-dim harmonic oscillator with spin-orbit interaction is

$$\begin{aligned} {}^q H &= \hbar\omega_0 \left\{ X_0^{(q)} - \kappa[2] \mathcal{L}^{(q)} \cdot \mathcal{S}^{(1/q)} \right\} = \\ &= \hbar\omega_0 \left\{ [N]q^{N+1} - \frac{q(q - q^{-1})}{[2]} C_{2,L}^{(q)} - \kappa \left(C_{2,J}^{(q)} - C_{2,L}^{(q)} - C_{2,S}^{(q)} - \frac{(q - q^{-1})^2}{[2]} C_{2,L}^{(q)} C_{2,S}^{(q)} \right) \right\} \end{aligned} \quad (47)$$

In (47) the factors have been chosen in accordance with the usual convention in classical (spherically symmetric) shell model with spin-orbit coupling. The eigenvectors of this Hamiltonian are given by (45) and the corresponding eigenvalues are

$$\begin{aligned} {}^q E_{n\ell j} &= \hbar\omega_0 \left\{ [n]q^{n+1} - \frac{q(q - q^{-1})}{[2]} [\ell][\ell + 1] \right. \\ &\quad \left. - \kappa \left([j][j + 1] - [\ell][\ell + 1] - [\frac{1}{2}][\frac{3}{2}] - \frac{(q - q^{-1})^2}{[2]} [\ell][\ell + 1][\frac{1}{2}][\frac{3}{2}] \right) \right\} \end{aligned} \quad (48)$$

In order to compare the expression for q -deformed $L - S$ interaction with the classical results we give the expansion of the expectation value (46) in series of τ , ($q = e^\tau$, $\tau \in \mathbf{R}$)

$$\begin{aligned} {}^q \langle n(\ell \frac{1}{2})jm | \mathcal{L}^{(q)} \cdot \mathcal{S}^{(1/q)} | n(\ell \frac{1}{2})jm \rangle_q &= \\ &= \begin{cases} \frac{1}{2} \ell \left(1 + \frac{\tau^2}{6} (4\ell^2 - 7) + \dots \right) & \text{if } j = \ell + 1/2 \\ -\frac{1}{2} (\ell + 1) \left(1 + \frac{\tau^2}{6} (4\ell^2 + 8\ell - 3) + \dots \right) & \text{if } j = \ell - 1/2 \end{cases} \end{aligned} \quad (49)$$

It can be easily seen that expression (46) introduces some corrections to the classical expressions for the spin-orbit interaction, which are proportional to τ^2 . These corrections

are small for the light nuclei, where the values of ℓ are small, but they are not negligible for the heavier nuclei, where the shells with bigger values of ℓ are important.

On the left side of figure 2 are shown the levels of MHO with spin-orbit term, considered again in the ^{208}Pb region [4] and the constants in the expression (1) $\mu' = k\mu$ and k are the same as in figure 1. On the right side of the same figure are shown the levels of the q -deformed 3-dimensional harmonic oscillator with q -deformed spin-orbit interaction. It should be noted that the levels of the q -deformed oscillator are calculated with the same values of the deformation parameter q for the oscillator term and the q -deformed $L - S$ term, and, as well as, with the same constant κ of the spin-orbit interaction for all shells.

5 Conclusions

The results of this paper demonstrate, that q -deformed algebras provide a natural background for the description of "deformable" physical systems.

In particular, we have shown, that in the enveloping algebra of $so_q(3)$ there exists an $so_q(3)$ scalar operator, built of irreducible vector operators according to the reduction $u_q(3) \supset so_q(3)$, for the case of the most symmetric representations of $u_q(3)$. In the limit $q \rightarrow 1$ this operator coincides with the Hamiltonian of the 3-dimensional harmonic oscillator, while, in the case of small deformations of the algebra, its spectrum reproduces the spectrum of the 3-dimensional modified harmonic oscillator (with, or without a spin-orbit interaction), making transparent the connection of our model with shell model calculations.

It should be emphasized, that in our case the parameters of the q -deformed model are the same for all shells, while in the modified harmonic oscillator the parameters vary from shell to shell.

The quantitative comparison of both models shows, that the q -deformation of the 3-dimensional harmonic oscillator effectively reproduces the "non-locality" and "deformations", introduced in the model of the modified harmonic oscillator by means of additional correction terms.

The application of the q -deformed 3-dimensional harmonic oscillator model to real nuclei will be considered in another paper, which is now in progress.

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